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GRAHAM'S PEBBLING CONJECTURE ON SHADOW GRAPH OF A PATH

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Abstract

Given a distribution of pebbles on the vertices of a connected graph G, a pebbling move is defined as the removal of two pebbles from some vertex and the placement of one of those pebbles at an adjacent vertex. The pebbling number, $f(G)$, of a connected graph G, is the smallest positive integer such that from every placement of $f(G)$ pebbles, we can move a pebble to any specified vertex by a sequence of pebbling moves. A graph G has the 2-pebbling property if for any distribution with more than $2f(G)-q$ pebbles, where q is the number of vertices with at least one pebble, it is possible, using the sequence of pebbling moves, to put 2 pebbles on any vertex. Graham conjectured that for any connected graphs G and H, $f(G\times H)\leq f(G)f(H)$. In this paper, we show that Graham's conjecture is true, when G is the shadow graph of a path and H is a graph having 2−pebbling property. Also we prove that Graham's conjecture is true for the product of two shadow graphs of paths.

Keywords: Pebbling number, 2-pebbling property, Shadow graph, Graham's conjecture.

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1. Introduction

Pebbling, one of the latest evolutions in graph theory proposed by Lakarias and Saks, has been the topic of vast investigation with significant observations. Having Chung [1] as the forerunner to familiarize pebbling into writings, many other authors too have developed this topic. Hulbert published a survey of graph pebbling [9]. For graph theoretic terminologies we refer to [5].

Consider a connected graph with fixed number of pebbles distributed on its vertices. A pebbling move consists of the removal of two pebbles from a vertex and placement of one of those pebbles at an adjacent vertex. The pebbling number of a vertex v in a graph G is the smallest number $f(G,v)$ such that for every placement of $f(G,v)$ pebbles, it is possible to move a pebble to v by a sequence of pebbling moves. Then the pebbling number of G is the smallest number, $f(G)$ such that from any distribution of $f(G)$ pebbles, it is possible to move a pebble to any specified target vertex by a sequence of pebbling moves. Thus $f(G)$ is the maximum value of $f(G,v)$ overall vertices v.

The pebbling number is known for many simple graphs including paths, cycles, and trees, but it is not known for most graphs and is hard to compute for any given graph that does not fall into one of these classes. Therefore, it is an interesting question if there is information we can gain about the pebbling number of more complex graphs from the knowledge of the pebbling number of some graphs for which we know. In the first paper on graph pebbling [1] Chung proposed the following conjecture. The conjecture is perhaps the most compelling open question in graph pebbling known as Graham's Conjecture.

Conjecture 1.1. ***(Graham [1])*** *For any connected graphs* G and H, *we have* $f(G\times H)\le f(G)f(H)$.

Chung [1] defined the 2-pebbling property of a graph. Given a distribution of pebbles on G, let p be the number of pebbles, q be the number of vertices with at least one pebble, we say that G satisfies the 2 - pebbling property, if it is possible to move two pebbles to any specified vertex whenever p and q satisfy the inequality $p+q>2f(G)$.

Chung [1] showed that complete graphs, trees and hypercubes have the 2-pebbling property and verified Conjecture 1.1. when G is a complete graph and H is a graph having the 2-pebbling property. The 2-pebbling property plays an important role in the study of Conjecture 1.1.

Moews [10] verified Conjecture 1.1 when G and H are trees and Snevily and Foster [4] extended this result to the case when G is a tree and H is a graph with 2-pebbling property. Wang et al. [13] verified this conjecture when G is a thorn graph of the complete graph and H is a graph with 2-pebbling property. Also the conjecture is verified when both G and H are fan graphs, wheel graphs [3] and when G is a complete bipartite graph and H is a graph with 2-pebbling property [2]. Likewise there are numerous articles [6], [7], [8], [12] that support Grahams conjecture.

In this paper, we show that the Conjecture 1.1. is true when G is the shadow graph of a path and H is any graph that satisfies the 2-pebbling property. In section 2, we introduce some definitions and notations which will be useful for the subsequent sections.

2. Preliminary

Definition 2.1. The *shadow graph* $D_2(G)$ of a connected graph G is constructed by taking two copies of G, say G_1 and G_2 and joining each vertex u in G_1 to the neighbours of the corresponding vertex v in G_2 .

The shadow graph of a path is denoted by $D_2(P_n)$. Label the vertices in the first copy of the path by $x_1, x_2, \ldots, x_n$ and the vertices in the second copy of the path by $x_{n+1}, x_{n+2}, \ldots, x_{2n}$ starting from the left.

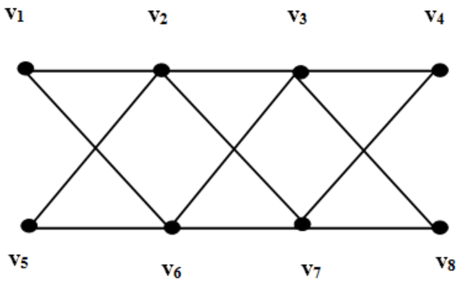

Figure 1.1. $D_2(P_n)$

Theorem 2.2. *[14] For the shadow graph of a path* P_n, $f(D_2(P_n)) = 2^{n-1} + 2$.

Theorem 2.3. *[4] Let* P_n *be a path on* n *vertices. Then* $f(P_n) = 2^{n-1}$.

Theorem 2.4. *[2] Let* $K_{1,n}$ *be a star graph, where* $n > 1$. *Then* $f(K_{1,n}) = n + 2$.

Theorem 2.5. *[11] (i). Let* C_{2k} *be an even cycle on* $2k$ *vertices. Then* $f(C_{2k}) = 2^k$.

(ii) Let C_{2k+1} *be an odd cycle on* $2k+1$ *vertices. Then*

$$f(C_{2k+1}) = 2\left\lfloor \frac{2^{k+1}}{3} \right\rfloor + 1.$$

Theorem 2.6. *[14] The graph $D_2(P_n)$ satisfies the two-pebbling property.*

Theorem 2.7. *[11] Let G be a graph with diameter G = 2. Then G has the 2- pebbling property.*

Theorem 2.8. *[4] All paths satisfy the 2-pebbling property.*

Theorem 2.9. *[4] All cycles have the 2-pebbling property.*

Theorem 2.10. *[3] The fan graph F_n satisfies the two pebbling property.*

Theorem 2.11. *[3] The star graph $K_{1,n}$ satisfies the two pebbling property*

3. The Graham's Conjecture

In this section we define the product of two graphs and discuss results on the pebbling number of direct product of the shadow graph of a path $D_2(P_n)$ and a graph satisfying 2-pebbling property.

Definition 3.1. [8] If $G=(V_G, E_G)$ and $H=(V_H, E_H)$ are two graphs, the direct product of G and H is the graph, $G \times H$, whose vertex set is the Cartesian product $V_{G\times H} = V_G \times V_H = \{(x, y) : x \in V_G, y \in V_H\}$ and whose edges are given by $E_{G\times H} = \{((x, y), (x', y')) : x = x' \textit{ and } (y, y') \in E_H \textit{ or } (x, x') \in E_G \textit{ and } y = y'\}$.

We can depict $G \times H$ pictorially by drawing a copy of H at every vertex of G and connecting each vertex in one copy of H

to the corresponding vertex in an adjacent copy of H. We write $\{x\}\times H$ (respectively, $G\times\{y\}$) for the subgraph of vertices whose projection onto V_G is the vertex x (respectively, whose projection onto V_H is y). If the vertices of G are labeled x_i then for any distribution of pebbles on $G\times H$, we write pi for the number of pebbles on $\{x_i\}\times H$ and qi for the number of occupied vertices of $\{x_i\}\times H$.

Lemma 3.2. *(Transfer Lemma) [8] Let (x_i, x_j) be an edge in G. Suppose that in $G\times H$, we have p_i pebbles occupying q_i vertices of $\{x_i\}\times H$, and r_i of these vertices have an odd number of pebbles. If $r_i \le k \le p_i$ and if k and p_i have the same parity then k pebbles can be retained on $\{\mathrm{a}_i\}\times H$ while moving $\frac{p_i - k}{2}$ pebbles onto $\{\mathrm{a}_j\}\times H$. If k and p_i have opposite parity we must leave k+1 pebbles on $\{\mathrm{a}_i\}\times H$, so we can move $\frac{p_i-(k+1)}{2}$ pebbles onto $\{\mathrm{a}_j\}\times H$. In particular we can always move at least $\frac{p_i - r_i}{2}$ pebbles onto $\{\mathrm{a}_j\}\times H$. In all these cases, the number of vertices of $\{\mathrm{a}_i\}\times H$ with an odd number of pebbles is unchanged by these movings.*

Theorem 3.3. *Let G be a graph which satisfies the 2 - pebbling property. Then $f(D_2(P_n)\times G) \le (2^{n-1}+2)f(G)$.*

Proof. Let $D_2(P_n): x_1, x_2, \ldots, x_{2n}$. We prove this theorem by induction on n. Since $D_2(P_2)$ is isomorphic to C_4, by [6] the result is true for n = 2. Assume the result is true for $3 \le n' < n$.

Let D be any distribution of $(2^{n-1}+2)f(G)$ pebbles on the vertices of $D_2(P_n)\times G$. Let (x_1, y) be the target vertex, where y is in G. By Transfer Lemma, we can transfer $\frac{p_n - q_n}{2}$ pebbles from $\{x_n\}\times G$ to $\{x_{n-1}\}\times G$ and also $\frac{p_{2n} - q_{2n}}{2}$ pebbles from $\{x_{2n}\}\times G$ to $\{x_{2n-1}\}\times G$. If

$$p_1 + p_2 + \ldots + p_{n-1} + \frac{p_n - q_n}{2} + p_{n+1} + p_{n+2} + \ldots + p_{2n-1} + \frac{p_{2n} - q_{2n}}{2}$$

$$\ge (2^{n-2} + 2)\, f(G$$

then we can use the induction to put $f(G)$ pebbles on $\{x_1\}\times G$ and hence we reach the target.

Also, since G satisfies 2−pebbling property, if

$$\frac{p_n - q_n}{2} > 2^{n-3} f(G) \text{ and } \frac{p_{2n} - q_{2n}}{2} > 2^{n-3} f(G),$$

then we can put 2^{n-2} pebbles on (x_n, y) and 2^{n-2} pebbles on (x_{2n}, y). Thus we can move 2^{n-3} pebbles from (x_n, y) to (x_{n-1}, y) and 2^{n-3} pebbles from (x_{2n}, y) to (x_{n-1}, y). Then (x_{n-1}, y) contains at least 2^{n-2} pebbles and hence we can reach the target using the path P_{n-2}.

Hence the only distributions from which we cannot reach the target (x_1, y) satisfy the inequalities

$$p_1 + p_2 + \ldots + p_{n-1} + \frac{p_n - q_n}{2} + p_{n+1} + p_{n+2} + \ldots + p_{2n-1} + \frac{p_{2n} - q_{2n}}{2} < (2^{n-2} + 2)\,\mathrm{f(G)}$$

$$\text{and } \frac{p_n - q_n}{2} + \frac{p_{2n} - q_{2n}}{2} \leq 2^{n-2} f(G).$$

But adding these inequalities together we get,

$$p_1 + p_2 + \ldots + p_{n-1} + p_n + p_{n+1} + p_{n+2} + \ldots + p_{2n-1} + p_{2n} < (2^{n-1} + 2)\,\mathrm{f(G)}.$$

Thus some configuration of pebbles from which we cannot pebble (x_1, y) must begin with fewer than $(2^{n-1} + 2) f(G)$ pebbles. By symmetry, we can pebble $(x_n, y), (x_{n+1}, y)$ and (x_{2n}, y) using any configuration of $(2^{n-1} + 2) f(G)$ pebbles on $D_2(P_n) \times G$.

Let (x_2, y) be the target vertex. Since G satisfies the 2−pebbling property, if $\frac{p_1 - q_1}{2} > f(G)$, we can reach the target. Also if $\frac{p_{n+1} - q_{n+1}}{2} > f(G)$, we can reach the target. By Transfer Lemma, we can move $\frac{p_1 - q_1}{2}$ pebbles to (x_2, y) and $\frac{p_{n+1} - q_{n+1}}{2}$ pebbles to (x_{n+2}, y). If

$$\frac{p_1 - q_1}{2} + p_2 + \ldots + p_n + \frac{p_{n+1} - q_{n+1}}{2} + p_{n+2} + \ldots + p_{2n-1} + p_{2n} \geq (2^{n-2} + 2)\,\mathrm{f(G)},$$

then we can use induction to move $f(G)$ pebbles on $\{x_2\} \times G$.

Hence the only distributions from which we cannot pebble the target (x_2, y) satisfy the inequalities

$$\frac{p_1 - q_1}{2} \le f(G); \quad \frac{p_{n+1} - q_{n+1}}{2} \le f(G) \text{ and}$$

$$\frac{p_1 - q_1}{2} + p_2 + \ldots + p_n + \frac{p_{n+1} - q_{n+1}}{2} + p_{n+2} + \ldots + p_{2n-1} + p_{2n} < (2^{n-2} + 2)\,\mathrm{f(G)}.$$

But adding these inequalities together we get,

$$p_1 + p_2 + \ldots + p_n + p_{n+1} + p_{n+2} + \ldots + p_{2n-1} + p_{2n} < (2^{n-1} + 2)\,\mathrm{f(G)}.$$

Thus some configuration of pebbles from which we cannot pebble (x_2, y) must begin with fewer than $(2^{n-1} + 2) f(G)$ pebbles.

By symmetry, we can pebble $(x_{n-1}, y), (x_{n+2}, y)$ and (x_{2n-1}, y) using any configuration of $(2^{n-1} + 2) f(G)$ pebbles on $D_2(P_n) \times G$.

Now let (x_i, y) be the target vertex, where $i \in \{3, 4, \ldots, n-2, n+3, n+4, \ldots, 2n-2\}$. Then there are at least $(2^{n-3} + 2) f(G)$ pebbles on $< D_2(P_n) \times G - [\{x_{n-1}\} \times G, \{x_n\} \times G, \{x_{2n-1}\} \times G, \{x_{2n}\} \times G] >$. Otherwise at least $(2^{n-3} + 2) f(G)$ pebbles are distributed on $< D_2(P_n) \times G - [\{x_1\} \times G, \{x_2\} \times G, \{x_{n+1}\} \times G, \{x_{n+2}\} \times G] >$. Thus by induction we can reach the target vertex.

Corollary 3.4. *Let P_n be a path on n vertices and $D_2(P_m)$ is a shadow graph of a path of length m, then* $f(D_2(P_m)\times P_n)\leq f(D_2(P_m))f(P_n)$.

Proof. The corollary follows from Theorem 2.8. and from Theorem 3.3.

Corollary 3.5. *Let C_n be a cycle on n vertices and $D_2(P_m)$ is a shadow graph of a path of length m, then* $f(D_2(P_m)\times C_n)\leq f(D_2(P_m))f(C_n)$.

Proof. The corollary follows from Theorem 2.9. and from Theorem 3.3.

Corollary 3.6. *Let F_n be a fan graph on n vertices and $D_2(P_m)$ is a shadow graph of a path of length m, then* $f(D_2(P_m)\times F_n)\leq f(D_2(P_m))f(F_n)$.

Proof. The corollary follows from Theorem 2.10. and from Theorem 3.3.

Corollary 3.7. *Let $K_{1,n}$ be a path on n+1 vertices and $D_2(P_m)$ is a shadow graph of a path of length m, then* $f(D_2(P_m)\times K_{1,n})\leq f(D_2(P_m))f(K_{1,n})$.

Proof. The corollary follows from Theorem 2.11. and from Theorem 3.3.

Theorem 3.8. $f(D_2(P_n)\times D_2(P_m))\leq f(D_2(P_n))f(D_2(P_m))$, *for any shadow graph of paths* $D_2(P_n)$ *and* $D_2(P_m)$.

Proof. Since the shadow graph of a path satisfies the 2-pebbling property and by Theorem 3.3, we conclude that, $f(D_2(P_n)\times D_2(P_m)) \le f(D_2(P_n))f(D_2(P_m))$.

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